Journal of Statistical Physics, Vol. 88, Nos. 5/6, 1997

On the Equivalence of the Parallel Channel and the Correlated Cluster Relaxation Models

Karina Weron[1] **and Marcin Kotulski**[2]

Received November 22, 1996

The question of the origins of nonexponential relaxation is addressed in terms of the probabilistic approach to relaxation. The interconnection between two differently rooted probabilistic models, i.e., between the parallel channel and the correlated cluster models, is presented. We show that clearly different probabilistic origins yield in both approaches a well-defined class of universally valid two-power-law responses with the stretched-exponential and exponential decay laws as special cases. The equivalence of both models indicates that variations in the local environment of the relaxing configurational units (parallel channel relaxation) can provide a basis for self-similar relaxation dynamics without the need for hierarchically constrained dynamics (correlated clusters relaxation).

KEY WORDS: Universal relaxation law; general relaxation equation; Lévy-stable distribution; Mittag-Leffler distribution; Burr distribution.

1. INTRODUCTION

Empirical evidence has accumulated over the years showing that the time-dependent change of macroscopic properties of physical systems evolving to equilibrium exhibits a great degree of universality.[1] The empirical data, including data from mechanical, dielectric and magnetic relaxation, NMR, dynamic light and quasielastic neutron scattering, as well as from reaction kinetics[1-6] show that there seems to be a universal decay (relaxation) function $\Phi(t)$ that relaxations of nonequilibrium states obey. On the basis

[1] Institute of Physics, Technical University of Wrocław, Wyspiańskiego 27, 50-370 Wrocław, Poland.

[2] Hugo Steinhaus Center for Stochastic Methods, Technical University of Wrocław, Wyspiańskiego 27, 50-370 Wrocław, Poland; e-mail: kotulski@im.pwr.wroc.pl; http://www.im.pwr.wroc.pl/~kotulski.

0022-4715/97/0900-1241$12.50/0

of linear dielectric response measurements, which have the important facility of allowing one to follow the regression of spontaneous structural fluctuations over several decades of time, the existence of the two-power-law response

$$-\frac{d\Phi(t)}{dt} \cong \begin{cases} (At)^{-n} & \text{for} \quad At \ll 1 \\ (At)^{-m-1} & \text{for} \quad At \gg 1 \end{cases} \tag{1}$$

in relaxation dynamics has been established unambiguously.[1] The parameters $0 < n$, $m \leqslant 1$ and $A > 0$ are the characteristic material constants. In some cases, however, the regression to equilibrium appears to be better described by the stretched-exponential decay law[7]

$$\Phi(t) = \exp[-(At)^{1-n}] \tag{2}$$

which is in accordance with (1) at short times only. Although, the stretched-exponential decay law is not universally valid[1,8] it appears frequently enough to call attention for the origins of its ubiquity and its relationship with the more general form (1) of the relaxation response.

The wide occurence of the two-power-law relaxation response, independently of the particular material property under consideration, attracts much theoretical attention for the underlying reasons of this phenomenon. It is well known that any relaxation process results from an appropriate transitional configuration into the system, imposed by nonequilibrium constraints at the time $t = 0$, and is conditioned by specific interactions of different parts of the system.[1,9-11] Both, the initial nonequilibrium state of a complex system and the internal interaction have, in general, random characteristics. Hence, the question of the origins of the universal relaxation law has to be addressed in terms of probabilistic models which can provide a clue to a better understanding of the physical mechanism of the relaxation. Such models, even with their inability to assign precise physical meaning to some variates give strict constrains on the mathematical form of the relaxation function[11] and the practical value of rigorous results in clarifying the nature of relaxation cannot be denied.

A common practice, following the historically oldest probabilistic attempt to relaxation,[12] has been to assign the nonexponentiality of relaxation to the distribution $F(b)$ of random effective relaxation (transition) rate $\tilde{\beta}$ by assuming[1-6,12-20] that the relaxation function $\Phi(t)$ is a weighted average of exponential decay with respect to the relaxation rate distribution

$$\Phi(t) = \int_0^\infty e^{-bt}\, dF(b) \tag{3}$$

In the specific case of the stretched-exponential law (12) the connection with Lévy-stable relaxation rate distribution has been known for over a decade[15–17] but without direct relation to the underlying stochastic mechanism. This deficiency has been removed by a more general probabilistic approach[21, 22] to the correlated clusters relaxation mechanism.[23] A microscopic stochastic scheme that uniquely leads to the macroscopic two-power-law response (1) has been proposed. In this probabilistic approach to the correlated clusters relaxation mechanism the hierarchical scheme of relaxation, by neglecting the inter-cluster correlations, is equivalent to the parallel channel relaxation[15–20] and the special case of the stretched-exponential law (2) results straightforwardly. This is due to the fact that in a noncorrelated clusters system the individually relaxing configurational units are characterized by independent and identically distributed (iid) random rates β_i only. The same result is obtained within the framework of the parallel channel model where relaxation is due to events occurring through a collection of independent channels. Since transformation of complex physical systems can be realized in several independent ways, the transition probability per unit time $\tilde{\beta}$ for the system as a whole is equal to a normalized sum of probabilities per unit time β_i over all the possible routes for its realization.[9] This fact appears to be a clue to a better understanding of the stochastic origins of the "universal" relaxation laws.[21, 22] Namely, if the non-negative rates β_i are assumed to be iid random variables, as it is the case of parallel channel relaxation models,[1–6, 15–20] the distribution $F(b)$ of the macroscopic (effective) rate $\tilde{\beta}$

$$\tilde{\beta} = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{\beta_i}{A_N} \tag{4}$$

where A_N is a sequence of suitable normalizing constants, takes the only possible form of the one-sided Lévy-stable distribution $S_\alpha(b)$, $0 < \alpha \leqslant 1$ (degenerate for $\alpha = 1$, i.e. $dS_\alpha(b)/db \to \delta(1)$ as $\alpha \to 1$, see Fig. 1). Its Laplace–Stieltjes transform (3) is of the stretched exponential form (2) with $\alpha = 1 - n$ (for a rigorous proof see ref. 22). The necessary and sufficient condition for the limit (4) to exist is that the distribution of the individual rates β_i belongs to the domain of attraction of the Lévy-stable law,[24, 25] i.e.,

$$\Pr(\beta_i \geqslant xb) \cong x^{-\alpha} \Pr(\beta_i \geqslant b)$$

for each $x > 0$, $0 < \alpha \leqslant 1$ and large b. In other words β_i have to be random variables with a common "broad" inverse power law distribution.[20] If β_i is the Lévy-stable variable itself, then the more restrictive stability condition

$$\tilde{\beta} = \sum_{i=1}^{N} \frac{\beta_i}{A_N} \tag{5}$$

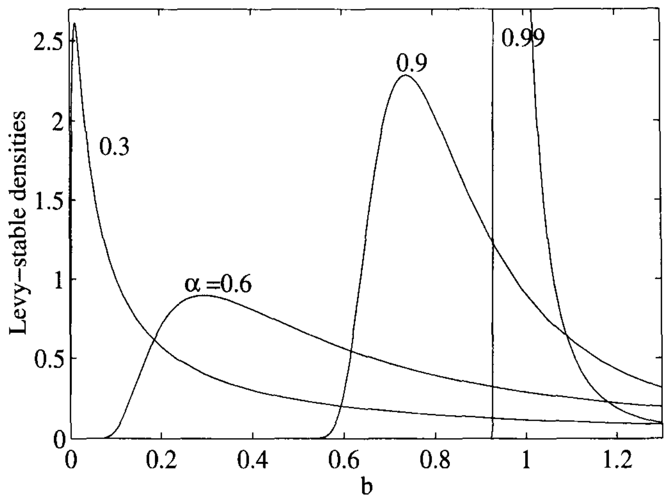

Fig. 1. One-sided Lévy-stable probability densities.

is fulfilled. Let us stress that condition (5) yields the equivalence of two different interpretations of the relaxation function,[26, 27] i.e., either as a function arising from an average over a lot of possible constant transition rates or as a function associated with a single age-dependent rate.

It is evident, that assumption (4) makes the parallel channel relaxation model useless in the general case when the universally valid class of two-power-law responses (1) is expected. Thus, it is natural to ask: under what conditions does the parallel channel model lead, if at all, to the two-power-law response?

The motivation of this paper is to show that two differently rooted probabilistic approaches to relaxation, namely the model of individually relaxing configurational units (parallel channel relaxation) and the correlated clusters model (hierarchical or serial relaxation) are equivalent. Both models, from clearly different mathematical reasons which have to be clarified on physical grounds, provide a well-defined class of two-power-law responses (1) with the stretched-exponential law (2) as a special case. Here we derive the explicit form of the effective relaxation rate distribution $F(b)$, which appears to be a generalization of the Mittag–Leffler distribution,[28, 29] under which the equivalence holds. The probabilistic origins of the Mittag–Leffler distribution indicate that, in order to derive the two-power-law response (1) in the framework of parallel channel models, the effective relaxation rate cannot be a result of the simple "deterministic" summation (4) of iid microscopic relaxation rates β_i. For this the random scheme of summation of iid random variables is required. Such a situation can be motivated by variations in the local environment of the relaxing configurational units in inhomogeneous systems.

The paper is organized as follows: in Section 2.1 we recall the basic idea of the probabilistic correlated clusters relaxation model,[21, 22, 26, 30–32] in Section 2.2 we analyze tail properties of the approximate solution of the relaxation equation derived in the framework of the correlated clusters mechanism and show its close relationship to the Burr distribution.[33] In Section 3 using the results of 2.1 we rigorously derive the explicit form of the effective relaxation rate distribution $F(b)$ which in the framework of the parallel channel model yields all empirically found relaxation responses. The probabilistic origins of the two-power-law and the stretched-exponential response in both approaches are, for convenience, presented in Table I of Section 4.

2. CORRELATED CLUSTERS MODEL

2.1. Rigorous Solution

Considering relaxation phenomena we deal with complex systems consisting of a large number N of subsystems (clusters) capable of changing their initial nonequilibrium states. The evolution of a system toward equilibrium is represented by the relaxation function which has to determine the probability $\Pr(\tilde{\theta} \geqslant t)$ that a transition of the system as a whole from its initial nonequilibrium state does not happen prior to a time instant t ($\tilde{\theta}$ denotes the life-time of the initial state).[9, 11] Explicitly, this probability is given by the first passage relaxation function $\Phi(t)$ defined as follows

$$\Phi(t) = \lim_{N \to \infty} \Pr[A_N \min(\theta_{1N},..., \theta_{NN}) \geqslant t] \tag{6}$$

where A_N is a sequence of suitable normalizing constants and θ_{iN}, $1 \leqslant i \leqslant N$, is the life-time of the ith subsystem in its initial state imposed by nonequilibrium constraints at $t = 0$.[21, 22, 26] The non-negative variables $\theta_{1N},..., \theta_{NN}$ are assumed to form a sequence of iid random variables for each N. Since every physical transformation process is conditioned by the interactions of different parts of the system (the causality principle[9, 14, 23]), the probability of preservation the initial state until a time instant t for each subsystem must necessarily be conditioned by appropriate variates reflecting the influence of the random intra- and inter-cluster dynamics. Thus, in order to model the physical reality in complex condensed matter systems, it is natural to assume for the initial nonequilibrium state of each subsystem the following Conditionally Exponential Decay (CED) property

$$\Pr(\theta_{iN} \geqslant t \mid \beta_i = b, \tilde{\eta}_i = s) = \begin{cases} 1 & \text{for } t = 0 \\ \exp(-bt) & \text{for } t < s \\ \exp(-bs) & \text{for } t \geqslant s \end{cases} \tag{7}$$

instead of a strictly exponential decay reflecting the intra-cluster dynamics only (for more details see[21, 22, 34]). The random transition rate β_i reflects the correlated intra-cluster dynamics. The variate $\tilde{\eta}_i = a_N^{-1} \max(\eta_1,..., \eta_{i-1}, \eta_{i+1},..., \eta_N)$, where a_N is a sequence of suitable normalizing constants and η_j denotes the time needed for the structural transformation of the jth subsystem evolving to equilibrium, is the stopping time reflecting the influence of the inter-cluster random correlations. The CED property determines the minimal probability $\exp(-bs) = const \in [0, 1)$ of preservation of the initial state for an individual ith subsystem until an infinitely long time. Because of the internal interactions, it is a physically acceptable assumption.[11, 14, 16] It simply reflects, in terms of the probabilistic language, the energetically unstable "hilltop" position[10] of an evolving to equilibrium subsystem. The non-negative variables $\beta_1, \beta_2,...$ and $\eta_1, \eta_2,...$ are assumed to form independent sequences of iid random variables. Let us note that the physical causality principle, indicated by the stochastic dependence of each θ_{iN} on β_i and $(\eta_1,..., \eta_{i-1}, \eta_{i+1},..., \eta_N)$, yields an *improper* distribution function for each θ_{iN}, i.e., a distribution function F such that $F(+\infty) = const < 1$.

Under the CED condition (7), the first passage relaxation function (6), if assumed $\Phi(\infty) = 0$, fulfils the following General Relaxation Equation

$$\frac{d\Phi(t)}{dt} = -\alpha A(At)^{\alpha-1} \left[1 - \exp\left(-\frac{1}{k(At)^{\alpha}} \right) \right] \Phi(t) \tag{8}$$

where $0 < \alpha \leqslant 1$, $A > 0$ and $k > 0$ (the detailed derivation of (8) is given in ref. 22). The solution of (8) can be expressed in the integral form only

$$\Phi(\mathrm{t}) = \exp\left\{ -\frac{1}{k} \int_0^{k(At)^{\alpha}} [1 - \exp(-x^{-1})] \, dx \right\} \tag{9}$$

and yields[22, 30, 31] the demanded power law properties (1) of the response function $f(x) = -d\Phi(t)/dt$ for $n = 1 - \alpha$, $m = \alpha/k$ and $k \geqslant \alpha$,

$$f(t) = \begin{cases} \alpha A(At)^{\alpha-1} & \text{for } At \ll 1 \\ \alpha A k^{-1-1/k} e^{-(1-\gamma_E)/k} (At)^{-\alpha/k-1} & \text{for } At \gg 1 \end{cases} \tag{10}$$

where $\gamma_E \approx 0.577216$ is the Euler constant.

Let us emphasize the special role of the parameter k in (8). It appears as a parameter in the sequence of normalizing constants in the max-stable law[22, 36] of type II. In order to incorporate the inter-cluster influence in the probabilistic representation (7), yielding in a natural way the special noncorrelated cluster case, the sequence of normalizing constants a_N in the expression

$$\tilde{\eta}_i = a_N^{-1} \max(\eta_1, ..., \eta_{i-1}, \eta_{i+1}, ..., \eta_N)$$

has to be of the following form

$$a_N = aN^{1/\alpha} \inf \left\{ t\colon F_\eta(t) \geqslant 1 - \frac{1}{N-1} \right\}$$

where F_η is the common distribution function of the sequence of random variables $\{\eta_i\}$ and $a^\alpha = kA^\alpha$. Observe that $k \to 0$ yields the solution of the stretched-exponential form (2). This corresponds to the case when inter-cluster influences, expressed by random variables $\tilde{\eta}_i$ are neglected. In this case $\exp(-bs) \to 0$ and the probability of preservation of the initial state in the subsystem, i.e. expression (7) takes strictly exponential form conditioned only by the value b of the random variable β_i. As a consequence the first passage relaxation function (6) simply equals[21, 22]: $\Phi(t) = \int_0^\infty e^{-bt}\, dS_\alpha(b)$. Parameter $k > 0$ indicates the slowing down influence of the inter-cluster correlations. Figure 2 represents the special cases of relaxation responses based on the solution (9) in terms of the values of the parameters k and α.

For practical purposes the solution (9) can be rewritten in the following form

$$\Phi(t) = \exp\left\{ -(At)^\alpha \left[1 - \exp\left(\frac{-1}{k(At)^\alpha} \right) \right] - \frac{1}{k} \Gamma\left(0, \frac{1}{k(At)^\alpha} \right) \right\}$$

where $\Gamma(a, z)$ is the incomplete gamma function (see ref. 35, Eq. (6.5.3), p. 260) defined as

$$\Gamma(a, z) = \int_z^\infty x^{a-1} e^{-x}\, dx$$

2.2. Approximate Solution

The relaxation function $\Phi(t)$ defined as the survival probability $\Pr(\tilde{\theta} \geqslant t)$ yields the existence of the probability distribution $F(t) = 1 - \Phi(t)$

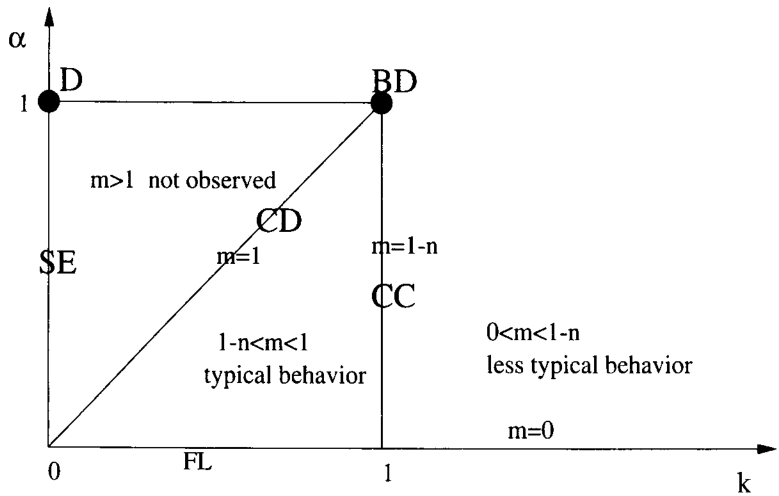

Fig. 2. A schematic representation of the special cases of the relaxation response based on Eq. (9) in terms of the values of the parameter k and of the exponent $\alpha = 1 - n$. The values of the exponent $m = \alpha/k$ are indicated. The abbreviations have the following meanings: SE—stretched exponential; D—Debye (exponential); BD—broadened Debye; CC—Cole-Cole; CD—Cole-Davidson; FL—flat loss.

of the random variable $\tilde{\theta}$ representing the life-time of the system as a whole in its initial nonequilibrium state. Unfortunately, the form (9) of the first passage relaxation function (6), derived under the CED condition, cannot be identified with any commonly known probability distribution. This is in contrast to the properties of the extreme value distributions which deal with sequences of iid random variables with a *proper* distribution function and the limiting distributions are of the well-known forms.[36] In the presented approach it will be the case of noncorrelated cluster systems, i.e., when instead of (7) we have the strictly exponential decay $\Pr(\Theta_{i,N} \geqslant t \mid \beta_i = b) = \exp(-bt)$. In order to find a commonly known distribution, closest to the form (9), let us derive the approximate solution of (9). Taking into account two terms in the series expansion of the exponential term $\exp(x^{-1})$ in the integrand of (9) we get the approximate form $\Phi_a(t)$ of the relaxation function equal to

$$\Phi_a(t) = \frac{1}{[1 + k(At)^{\alpha}]^{1/k}}, \qquad 0 < \alpha \leqslant 1, \quad k \geqslant \alpha \tag{11}$$

This simple analytical form of the relaxation function exhibits the following properties:

(i) The time derivative of $\Phi_a(t)$ has the two-power-law property which differs from (10) only in the long time term by the factor $\exp[-(1-\gamma_E)/k]$, namely

$$\frac{f(t)}{f_a(t)} \cong \begin{cases} 1 & \text{for} \quad At \ll 1 \\ \exp[-(1-\gamma_E)/k] & \text{for} \quad At \gg 1 \end{cases} \tag{12}$$

Figure 3 gives the plot of the long time quotient in (12). Figure 4 presents, for given α and k, the response function obtained from the rigorous (9) and approximate (11) forms of the relaxation function.

(ii) As $k \to 0$, $\Phi_a(t)$ takes the expected stretched exponential form

$$\Phi_a(t) = \frac{1}{[1+k(At)^\alpha]^{1/k}} \xrightarrow{k \to 0} \exp[-(At)^\alpha]$$

(iii) The approximate form (11) can be identified with a special case of the Burr distribution[33] which we denote by $B_{b,c}(x) = 1-(1+x^b)^{-c}$, $x > 0$,

$$\Phi_a(t) = 1 - B_{\alpha, 1/k}(k^{1/\alpha} At)$$

Note, that the particular form of the Burr distribution $B_{\alpha, 1/k}(k^{1/\alpha} At)$, as $k \to 0$, tends to the Weibull distribution $W_\alpha(t) = 1 - \exp[-(At)^\alpha]$, $0 < \alpha \leqslant 1$. At this point, we also have to stress the main differences in properties of

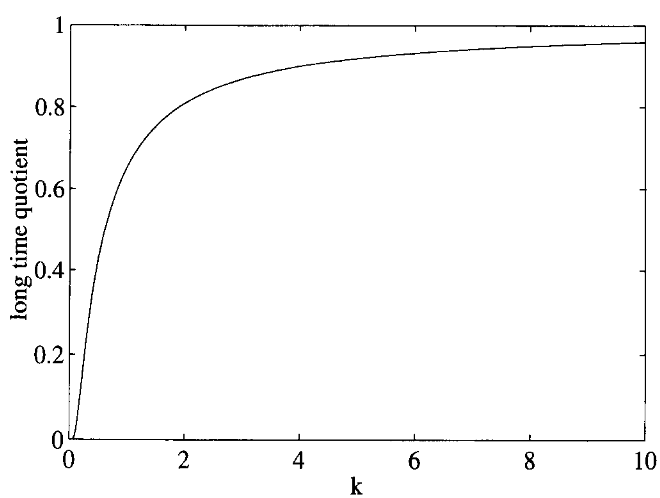

Fig. 3. The plot of long time quotient $\exp[-(1-\gamma_E)/k]$ in Eq. (12).

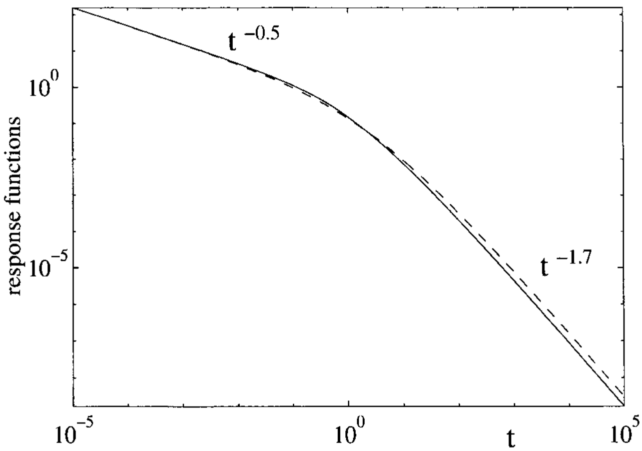

Fig. 4. The rigorous (solid line) and approximate (dashed line) response functions for $\alpha = 0.5$ and $k = 0.7$.

both distributions. Namely, random variate $\tilde{\theta}$ distributed with the Weibull distribution has finite expected value $\langle \tilde{\theta} \rangle < \infty$ for all α, while for the Burr distribution only if $\alpha/k > 1$. The empirical data (1) indicating the range (0, 1] for the power laws exponents n and m, "choose" the heavy tailed Burr life-time distribution with $\alpha/k \leqslant 1$ for which $\langle \tilde{\theta} \rangle = \infty$.

3. PARALLEL CHANNEL RELAXATION

3.1. Power Laws Response

The question of the origins of nonexponential relaxation addressed in terms of the parallel channel models[1-6, 13-20] is based on the distribution of random relaxation (transition) rates β_i. Each rate corresponds to one channel of relaxation and channels are assumed to operate in a parallel way, i.e., individual relaxation rates are independent. If the macroscopic (effective) relaxation rate $\tilde{\beta}$ is obtained as a normalized sum, Eq. (4), of individual relaxation rates β_i the stretched-exponential form (2) of the Laplace–Stieltjes transform (3) of the effective rate distribution $F(b)$ is the only nondegenerate form available.[21, 22, 24-26]

In order to obtain the more general class (1) of relaxation responses in the framework of the parallel channel model, instead of the "deterministic" (4), the so-called random summation of iid individual rates β_i should be considered.

Let us first recall that the Burr distribution can be obtained as a smooth mixture of the Weibull distribution compounded with respect to a random scale parameter distributed with the gamma distribution.[33] Using this fact, we can rewrite (11) in the following form

$$\Phi_a(t) = \int_0^\infty \exp(-\lambda^\alpha t^\alpha)\, d\Gamma_{1/k,\,k}(\lambda^\alpha/A^\alpha), \tag{13}$$

where

$$\Gamma_{1/k,\,k}\left(\frac{\lambda^\alpha}{A^\alpha}\right) = \frac{1}{\Gamma(1/k)} \int_0^{\lambda^\alpha/A^\alpha} \left(\frac{x}{k}\right)^{1/k-1} e^{-x/k}\, d\left(\frac{x}{k}\right) \tag{14}$$

is the generalized gamma distribution and λ is the randomized scale parameter in the Weibull distribution. The stretched-exponential term in the integrand of (13), by relation (3), equals

$$\exp(-\lambda^\alpha t^\alpha) = \int_0^\infty e^{-bt}\, dS_\alpha(b/\lambda) \tag{15}$$

where $S_\alpha(b/\lambda)$ is the one-sided Lévy-stable distribution of the effective rate $\tilde{\beta}$ resulting from the summation (4) of iid β_i. Substituting (15) into (13), integrating by parts and using the properties of a one-sided distribution function ($F(0) = 0$, $F(\infty) = 1$), we get

$$\Phi_a(t) = \int_0^\infty e^{-bt} S_\alpha(b/\lambda)\, d\Gamma_{1/k,k}(\lambda^\alpha/A^\alpha) = \int_0^\infty e^{-bt}\, dF_{\alpha,\,1/k}\left(\frac{b}{k^{1/\alpha}A}\right) \tag{16}$$

where

$$F_{\alpha,\,1/k}\left(\frac{b}{k^{1/\alpha}A}\right) = \int_0^\infty S_\alpha(b/\lambda)\, d\Gamma_{1/k,\,k}(\lambda^\alpha/A^\alpha) \tag{17}$$

$0 < \alpha \leqslant 1$, $k \geqslant \alpha$, is the distribution of the effective relaxation rate $\tilde{\beta}$ obtained by randomizing the scale parameter λ in $S_\alpha(b/\lambda)$ with the generalized gamma distribution (14). The one-sided probability distribution with the Laplace–Stieltjes transform of the form (11) has the following series representation

$$F_{\alpha,\,1/k}\left(\frac{b}{k^{1/\alpha}A}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1/k+n)}{n!\,\Gamma(1/k)\,\Gamma(1+\alpha/k+\alpha n)} \left(\frac{b}{k^{1/\alpha}A}\right)^{\alpha/k+\alpha n}$$

which is the generalized form of the Mittag–Leffler distribution introduced in ref. 28. Figure 5 shows the properties of the Mittag–Leffler probability

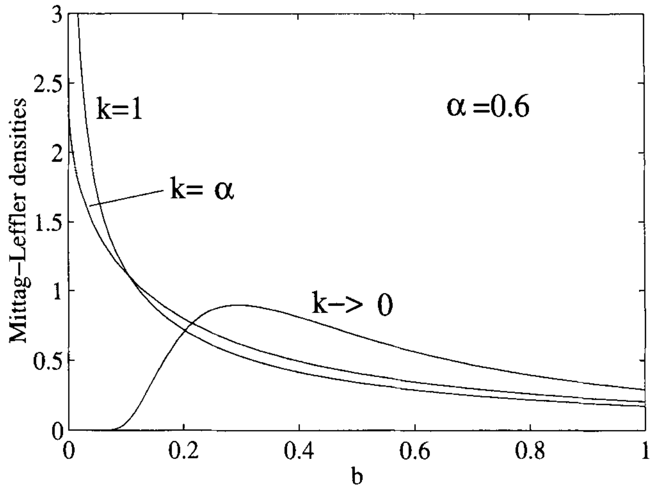

Fig. 5. Mittag–Leffler probability densities $dF_{\alpha,1/k}(b/[k^{1/\alpha}A])/db$ for $\alpha = 0.6$. Mittag–Leffler densities are monotone with an infinite peak for $k > \alpha$ and tend to unimodal Lévy-stable densities as $k \to 0$ for arbitrary $0 < \alpha < 1$.

densities for $k \geqslant \alpha$. Equation (17) yields the Lévy-stable density as k tends to 0 for arbitrary $0 < \alpha < 1$. In this case the gamma distribution takes the degenerate form.

3.2. Random Sums of Individual Relaxation Rates

It follows from Eq. (16) that in order to obtain the two-power-law response (1) in the framework of the parallel channel relaxation, the distribution $F(b)$ of the effective relaxation rate $\tilde{\beta}^*$ has to be of the form given by Eq. (17). This is equivalent to the statement that the effective relaxation rate $\tilde{\beta}^*$ is a *v-stable* random variable given by the following expression

$$\tilde{\beta}^* = (G_{1/k,\,k})^{1/\alpha}\, \tilde{\beta} \tag{18}$$

where $G_{1/k,\,k}$ is a gamma random variable and $\tilde{\beta}$ is a Lévy-stable (or α-stable) effective relaxation rate. (The representation of the distribution of v-stable random variables via Lévy-stable laws, as well as its tail properties has been studied recently[37, 38]). In general, a v-stable effective relaxation rate $\tilde{\beta}^*$ is obtained as a limit of normalized sums of a random number v_n of iid individual relaxation rates β_i

$$\tilde{\beta}^* = \lim_{n \to \infty} \frac{1}{a_n} \sum_{i=1}^{v_n} \beta_i \tag{19}$$

where a_n is a sequence of suitable normalizing constants and v_n is a random variable independent of β_i. Observe that if v_n in (19) takes a constant value n then $\tilde{\beta}^* = \tilde{\beta}$ is the Lévy-stable variable defined in (4). This corresponds to the degenerate gamma distribution of the random variable $(G_{1/k,\,k})^{1/\alpha}$ in (18), reached as $k \to 0$. In the particular case of the ν-stable $\tilde{\beta}^*$, given by Eq. (18), v_n has to be of the form

$$v_n = V_{1/n,\,1/k}, \qquad k > 0 \tag{20}$$

where $V_{1/n,\,1/k}$ is a negative binomial random variable,

$$\Pr(V_{1/n,\,1/k} = r) = \frac{\Gamma(1/k + r)}{\Gamma(1/k)\, r!} \left(\frac{1}{n}\right)^{1/k} \left(1 - \frac{1}{n}\right)^r, \qquad r = 0, 1, 2,...$$

such that

$$\frac{V_{1/n,\,1/k}}{n} \to G_{1/k,\,1}, \qquad \text{as} \quad n \to \infty$$

Since the distribution of the individual rates β_i belongs to the domain of attraction of the Lévy-stable law for $a_n = (kn)^{-1/\alpha}$

$$(kn)^{-1/\alpha} \sum_{i=1}^{n} \beta_i \to \tilde{\beta}, \qquad \text{as} \quad n \to \infty$$

we have, for $v_n = V_{1/n,\,1/k}$, that

$$n^{-1/\alpha} \sum_{n=1}^{v_n} \beta_i \xrightarrow{n \to \infty} (kG_{1/k,\,1})^{1/\alpha} \tilde{\beta} = (G_{1/k,\,k})^{1/\alpha} \tilde{\beta} = \tilde{\beta}^*$$

Thus, if relaxation is due to events occurring through a collection of a random number v_n of independent channels, being a negative binomial variate (20), the parallel channel relaxation mechanism leads to the "universal" response (1). Taking into account all possible ("deterministic" number) channels of relaxation in the system, only the stretched-exponential response (2) is available.

4. CONCLUSIONS

We have discussed the equivalence of two differently rooted probabilistic approaches to relaxation, namely of the parallel channel and the correlated clusters models. For the first time we have shown that the parallel channel model in a natural way can lead to the "universal" two-power-law response (1). This result indicates that the short- and long-time self-similar relaxation dynamics may originate not only from the sequentially constrained dynamics of the correlated clusters but also may be attributed to variations in the local environment of the independently relaxing configurational units. (Such a conclusion is in accordance with recent analysis of random transition rate mechanism for the stretched-exponential relaxation[20]). In both models, the probabilistic origins of the empirically found responses have a clear mathematical structure, see Table I. A unifying feature, leading in both approaches from the stretched exponential decay law to the class of two-power-law responses, is the passage from the commonly used tools of probability theory, i.e., from the "deterministic" summation of iid random variables and proper distribution functions, to more advanced techniques dealing with random sums of iid random variables and improper distribution functions. Such a qualitative change should be reflected in physical mechanisms underlying the power laws and the stretched-exponential relaxations. One should be able to differentiate the physical constraints underlying the notions of random sums of iid or deterministic sums of iid individual transition rates in the parallel channel model, as well as notions of proper or improper individual life-time distribution functions in the correlated cluster model. This assertion finds support in the frequency domain results,[39] where a gradual shift from serial relaxation at low frequencies to parallel relaxation at high frequencies has been found.

Table I. Probabilistic Origins of the "Universal" Relaxation Response in the Parallel Channel and the Correlated Clusters Relaxation Model

	Parallel channel relaxation	Correlated cluster relaxation
Definition of the relaxation function	$\Phi(t)=\int_0^\infty e^{-bt}\,dF_{\tilde\beta}(b)$	$\Phi(t)=\lim_{N\to\infty}\Pr(A_N\min[\theta_{1N},\ldots,\theta_{NN}]\geqslant t)$
Probabilistic origins of the stretched exponential response	Deterministic summation of iid individual relaxation rates β_i $\tilde\beta=\lim_{N\to\infty}\sum_{i=1}^{N}\beta_i/A_N$	Proper life-time distribution of an individual subsystem in its imposed initial state $\Pr(\theta_{iN}\geqslant t\mid\beta_i=b)=e^{-bt}$
Probabilistic origins of the power laws response	Random summation of idd individual relaxation rates β_i $\tilde\beta=\lim_{n\to\infty}\sum_{i=1}^{\nu_n}\beta_i/A_n$, where ν_n is neg. binomial r.v.	Improper life-time distribution of an individual subsystem in its imposed initial state (random cut-off time at $t=s$) $\Pr(\theta_{iN}\geqslant t\mid\beta_i=b,\tilde\eta_i=s)=e^{-b\min(t,s)}$

ACKNOWLEDGMENT

This work was supported in part by KBN Grant No. 2 P03B 100 13.

REFERENCES

1. A. K. Jonscher, *Universal Relaxation Law* (Chelsea Dielectrics Press, London, 1996).
2. T. V. Ramakrishnan and M. Raj Lakshmi, *Non-Debye Relaxation in Condensed Matter* (World Scientific, Singapore, 1987).
3. D. Richter, A. J. Dianoux, W. Petry, and J. Teixeira, *Dynamics of Disordered Materials* (Springer Verlag, Berlin, 1989).
4. J. Colmenero, *J. Non-Cryst. Solids* **131–133**:860 (1991).
5. J. Colmenero, A. Alegria, J. M. Alberdi, F. Alvarez, and B. Frick, *Phys. Rev. B* **44**:7321 (1991).
6. A. Plonka, *Prog. Reaction Kinetics* **16**:157 (1991); *Ann. Rep. Prog. Chem.* **89**:37 (1992).
7. G. Williams and D. C. Watts, *Trans. Faraday Soc.* **66**:80 (1970).
8. S. Havriliak Jr and S. J. Havriliak, *J. Non-Cryst. Solids* **172–174**:297 (1994).
9. L. Mayants, *The Enigma of Probability in Physics* (Reidel, Dordrecht, 1984).
10. G. Nicolis and I. Prigogine, *Exploring Complexity* (Freeman, New York, 1989).
11. J. Klafter and M. F. Shlesinger, *Proc. Natl. Acad. Sci. USA* **83**:848 (1986).
12. K. W. Wagner, *Ann. Phys.* **4–40**:817 (1913).
13. K. L. Ngai, *Comments Solid State Phys.* **9**:127 (1979); **9**:141 (1980).
14. R. G. Palmer, D. Stein, E. S. Abrahams, and P. W. Anderson, *Phys. Rev. Lett.* **53**:958 (1984).
15. E. W. Montroll and J. T. Bendler, *J. Stat. Phys.* **34**:129 (1984).
16. J. T. Bendler, *J. Stat. Phys.* **36**:625 (1984).
17. K. Weron, *Acta Phys. Pol. A* **70**:529 (1986).
18. I. M. Hodge, *J. Non-Cryst. Solids* **169**:211 (1994).
19. A. Hunt, *J. Non-Cryst. Solids* **168**:258 (1994).
20. I. Koponen, *J. Non-Cryst. Solids* **189**:154 (1995).
21. K. Weron, *J. Phys.: Condens. Matter* **3**:9151 (1991); **4**:10507 (1992).
22. K. Weron and A. Jurlewicz, *J. Phys. A.: Math. Gen.* **26**:395 (1993).
23. L. A. Dissado and R. M. Hill, *Proc. R. Soc. A* **390**:131 (1983).
24. V. M. Zolotarev, *One-dimensional Stable Distributions* (American Mathematical Society, Providence, 1986).
25. A. Janicki and A. Weron, *Simulation and Chaotic Behavior of α-Stable Stochastic Processes* (Marcel Dekker, New York, 1994).
26. A. Weron, K. Weron, and W. A. Woyczynski, *J. Stat. Phys.* **78**:1027 (1995).
27. V. Halpern, *J. Phys.: Condens. Matter* **7**:7687 (1995).
28. R. N. Pillai, *Ann. Inst. Statist. Math.* **42**:157 (1990).
29. K. Weron and M. Kotulski, *Physica A* **232**:180 (1996).
30. A. Jurlewicz and K. Weron, *J. Stat. Phys.* **73**:69 (1993).
31. A. Jurlewicz, *J. Stat. Phys.* **79**:993 (1995).
32. A. Jurlewicz, A. Weron, and K. Weron, *Appl. Math.* **23**:379 (1996).
33. L. Devroye, *Non-Uniform Random Variate Generation* (Springer, New York, 1986).
34. K. Weron, *J. Phys.: Condens. Matter* **4**:10503 (1992).
35. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

36. M. R. Leadbetter, G. Lindgren, and H. Rootzen, *Extremes and Related Properties of Random Sequences and Processes* (Springer, New York, 1986).
37. T. J. Kozubowski and A. K. Panorska, *Statistics Prob. Letters*, **29**:307 (1996).
38. T. J. Kozubowski and K. Podgórski, submitted.
39. A. Hunt, J. Phys.: *Condens. Matter* **6**:8087 (1994).